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Graph theory and relativistic field equations for half odd integer spin and unique mass

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Abstract. Graph theoretical methods are used to analyse relativistic field equations for half odd integer spin and unique mass. The analysis is easiest when repeated irreducible representations (RIR) of the Lorentz group do not occur, but the methods apply in general cases, and can also be used for equations with a mass–spin spectrum. A simple graph theoretical method for finding the possible minimal polynomials of L_0 is given, and some general results on the possible structure of the equations are obtained. As an example, all theories of spin-$\frac{5}{2}$, without RIR, are considered and it is shown that there are none with unique mass. Theories with RIR are briefly discussed.

1. Introduction

This paper is directed at the problem of practical evaluation of high-spin field theories based on the usual equation (2.1). While much is known about the theory of (2.1), its practical study in a particular case involves much complicated but elementary algebra (Capri 1969, Capri and Shamaly 1971, Hurley and Sudarshan 1974, 1975, Khalil 1977, Kobayashi 1977). In earlier papers we have given a graph theoretical approach which obviates some of this algebra (Cox 1974a, b, c). Graphs associated with (2.1) and a given reducible representation $\mathscr{R}$ of the proper Lorentz group $\mathscr{L}_{\mathrm{p}}$ are used to characterise the equations and analyse the mass–spin spectra. We concentrate on unique mass–spin theories, but the methods apply also to multi-mass–spin theories.

In the earlier papers the graphs were used to study the characteristic equations of the s-blocks of L_0, and to count the conditions required for quantisable theories. There was no guarantee that these conditions could be satisfied for a given representation $\mathscr{R}$, and although the general form of the minimal polynomial of L_0 was known, there was no way of obtaining this for a particular $\mathscr{R}$, other than by direct calculation. Here we show how the possible forms of the minimal polynomial of the s-blocks can be obtained immediately by visual inspection of their graphs. The results can be used to eliminate many possible mass–spin spectra without any calculation, or if calculation is needed, it is made most efficient. As an example we consider unique mass–spin theories without repeated irreducible representations (RIR) of $\mathscr{L}_{\mathrm{p}}$. We give the most general possible forms for the maximum spin blocks and show that there are no unique mass–spin theories for spin-$\frac{5}{2}$ unless RIR are introduced. We expect this to be so for any half odd integer spin greater than $\frac{3}{2}$.

We assume (2.1) is covariant under the complete Lorentz group and is derivable from a Lagrangian (actually the last requirement is not essential). Also, we consider half odd integer spin because the results seem to be stronger in this case. When RIR are present the graphical results (except theorem 3) still hold, but are more difficult to use.

The reason for studying (2.1) is that, because of the well known inconsistency problems of high-spin interacting fields, it is important to understand fully the structure of such equations. So far studies of (2.1) have been confined to simple representations $\mathcal{R}$ and we hope the methods given here will make more complicated representations accessible.

In § 2, equation (2.1) and its graphs are briefly reviewed. Section 3 contains the required graph theory results. Section 4 applies these results to unique mass–spin theories without RIR, while § 5 looks briefly at equations with RIR. Useful results are still obtainable in the latter case, in particular we are able to specify precisely the minimum polynomial of the recently proposed high-spin theory of Singh and Hagen (1974). Section 6 is the conclusion.

2. Field equations and associated graphs

Irreducible representations of $\mathcal{L}_p$ are denoted by $\tau = (l_0, l_1)$ where l_0, l_1 are both half odd integers and $l_1 > |l_0|$. The connection with the usual notation $\mathcal{D}(k, l)$ is

$$l_0 = k - l, \qquad l_1 = k + l + 1.$$

On restriction to the space rotation group (l_0, l_1) reduces to spin representations with weights $|l_0|, |l_0|+1, \ldots, l_1 - 1$. The representation conjugate to $\tau = (l_0, l_1)$ is denoted by $\dot{\tau} = (-l_0, l_1)$, and so space reflection corresponds to reflection in the l_1 axis.

The general finite-dimensional first-order free field equation for half odd integer spin can be written

$$(L_\mu \, \partial^\mu + \mathrm{i}\chi)\psi = 0 \tag{2.1}$$

where ψ transforms according to some in general reducible representation $\mathcal{R}$ of $\mathcal{L}_p$. Choosing matrix representations for L_μ and ψ (we take χ as a multiple of the unit matrix) such that the $\mathcal{L}_p$ representation is a direct sum of irreducible representations τ_i of $\mathcal{L}_p$, L_0 can be written in the form

$$L_0 = \textstyle\sum_{\oplus} A_s$$

where

$$A_s = [C_s^{\tau\tau'}]$$

and the spin index takes the values $s = |l_0|, |l_0|+1, \ldots, l_1 - 1$ for the irreducible representation $\tau \sim (l_0, l_1)$. The elements of the s-block, A_s, are scalar matrices and its dimension is given by the number of representations $(l_0^{(i)}, l_1^{(i)})$ such that $|l_0^{(i)}| \leq s \leq l_1^{(i)} - 1$.

The mass–spin spectra of the theory is provided by the eigenvalues of the s-blocks. In particular, if (2.1) is to describe a particle with unique rest mass m and spin j then all A_s must be nilpotent except for A_j, which must have exactly two non-zero eigenvalues $\pm\chi/m$.

The Lorentz covariance of (2.1) implies that the $C_s^{\tau\tau'}$ are non-zero only for coupled representations:

$$\begin{aligned}
&(l_0', l_1') = (l_0+1, l_1) \\
&\qquad C_s^{\tau\tau'} = \rho(s, l_0)C^{\tau\tau'} \\
&\qquad C_s^{\tau'\tau} = \rho(s, l_0)C^{\tau'\tau} \\
&(l_0', l_1') = (l_0, l_1+1) \\
&\qquad C_s^{\tau\tau'} = \rho(s, l_1)C^{\tau\tau'} \\
&\qquad C_s^{\tau'\tau} = \rho(s, l_1)C^{\tau'\tau}
\end{aligned} \tag{2.2}$$

where $C^{\tau\tau'}$ are arbitrary complex numbers and $\rho(s, n) = |[(s+n+1)(s-n)]^{1/2}|$. Space reflection covariance implies

$$C^{\tau\tau'} = C^{\tau^{\cdot}\tau'^{\cdot}}$$

which with the requirement of Lagrangian origin yields

$$C_s^{\tau'\tau} = s(\tau, \tau')\overline{C_s^{\tau\tau'}}$$

(cf integer spin case, Cox 1974a, b, c) where $s(\tau, \tau') = \pm 1$.

As in the integer spin case, a particular equation (2.1) is characterised by a linear graph G with vertices denoting the irreducible representations τ_i in $\mathscr{R}$ and an undirected edge connecting those representations τ, τ' such that $C_s^{\tau\tau'} \neq 0$ for some s.

Any theory based on (2.1) for half odd integer spin without RIR can be represented by a subgraph of a lattice graph exemplified in figure 1. If RIR occur a vertex is used for each copy of the same representation. In this case G is no longer a lattice graph.

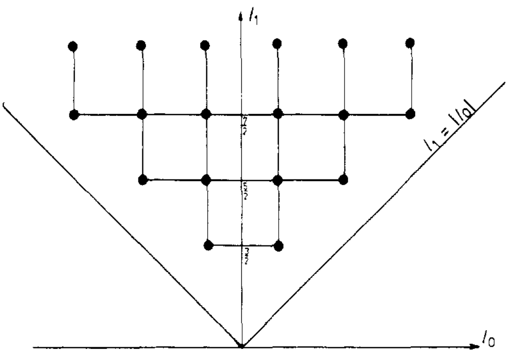


Figure 1. Lattice graph for a general maximum spin-$\frac{5}{2}$ theory without RIR.

The representations τ_i which can appear in an s-block, A_s, are those in or on the rectangle

$$\begin{aligned}
l_0 &= -s, & l_0 &= s \\
l_1 &= s+1, & l_1 &= j+1
\end{aligned}$$

where j is the maximum value of s in the representation R. This corresponds to a subgraph G_s of G. The corresponding directed graph (digraph) with directed edges in both directions between any two vertices adjacent in G_s, is denoted by D_s. The edges of D_s may be labelled with the corresponding elements of A_s, which may conversely be written down by reference to the graph D_s.

G and G_s are bigraphs (Cox 1974c) and so the s-blocks can be partitioned in the form

$$\begin{matrix} & V_1 & V_2 & \\ & \left[\begin{matrix} 0 \\ C_s \end{matrix}\right. & \left.\begin{matrix} B_s \\ 0 \end{matrix}\right] & \begin{matrix} V_1 \\ V_2 \end{matrix} \end{matrix}$$

V_1, V_2 being the usual disjoint vertex sets in G_s (see § 3). Since τ_i and $\dot{\tau}_i$ are connected by paths of odd length, they must lie in different vertex sets, thus if $\tau_i \in V_1$ then $\dot{\tau}_i \in V_2$. If we now arrange the rows and columns of A_s in the form

$$\begin{matrix} V_1 & & V_2 \\ \tau_1\tau_2 \ldots \tau_n & & \dot{\tau}_1\dot{\tau}_2 \ldots \dot{\tau}_n \end{matrix}$$

space reflection symmetry implies that the s-block takes the form

$$A_s = \begin{bmatrix} 0 & B_s \\ B_s & 0 \end{bmatrix}. \tag{2.3}$$

3. Graph theoretical results

Let G be a linear graph with n vertices $\{v_i\}$ and edges (v_i, v_j), and D the corresponding digraph obtained by replacing each edge of G by two undirected edges one in each direction.

An *edge sequence* or *walk* of length l in G is a finite sequence of edges of the form $(v_0, v_1), (v_1, v_2) \ldots (v_{l-1}, v_l)$. An edge sequence in which all the edges are distinct is called a *path* (Wilson 1972). If in addition all the vertices are distinct (except possibly, $v_0 = v_l$) then the path is a *chain*. A closed chain ($v_0 = v_l$) is a *cycle*.

A set of disjoint cycles (SDC) in D is a set of directed cycles (all edges like directed) in which no pair of cycles passes through the same vertex. A set of disjoint cycles in G has the same definition as for D without the reference to direction. However, the cycles of length two in D correspond to the *edges* of G, so any set of disjoint 2-cycles in D corresponds to a set of vertex disjoint edges in G, i.e. a *matching* in G. A matching in G containing the maximum possible number of edges is called a *maximal matching* (MM) on G, and the number of edges in a MM is called the *line independence number* of G, denoted $\beta_1(G)$. A matching which covers all vertices of a graph is called a *perfect matching* on the graph.

A bipartite graph (bigraph), $G(V_1, V_2)$, is one in which the vertex set V can be partitioned into two disjoint subsets V_1, V_2 such that no vertex in $V_1(V_2)$ is adjacent to any other vertex in $V_1(V_2)$. A graph is a bigraph if and only if all of its cycles are of even length.

An *associated matrix* of the digraph D, denoted $A(D)$, is a matrix which only has non-zero elements in positions i, j if there is a directed edge from v_i to v_j. The ijth element a_{ij} associated with this edge is called the *weight* of the edge. The non-zero elements of the *general associated matrix* are arbitrary and it merely forms a matrix representation of the connectivity of D. The associated matrices occurring in relativistic equations considered in this paper are quasi-Hermitian, i.e.

$$a_{ji} = s_{ij}\bar{a}_{ij}$$

where $s_{ij} = \pm 1$, although it is not always necessary to assume this to obtain useful information.

If D' is a subgraph of D, then the *edge weight product* over D' is the product of the weights of all the edges in D'. Let $c_r(D)$ denote the edge weight product over a SDC on D covering r vertices. If the SDC are numbered in some way, denote the edge weight product for the ith set by $c_r^{(i)}(D)$. Define

$$\mathscr{C}_r(D) = \sum (-1)^{l_i} c_r^{(i)}(D) \tag{3.1}$$

where l_i is the number of cycles with even length in the ith SDC. $\mathscr{C}_r(D)$ is then the coefficient of $(-\lambda)^{n-r}$ in the characteristic polynomial $\Delta(-\lambda)$ for $A(D)$ (Cox 1974a, b, c).

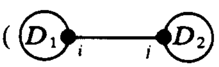
If we are only interested in the 2-cycles of D, for example if G is a tree, then it is convenient to consider G instead of D and to take the weight of the edge (i, j) to be $\omega_{ij} = a_{ij}a_{ji}$. Then if we denote the edge weight product of the ith matching on r vertices, in the graph G, by $M_{r/2}^{(i)}G$, we define

$$M_{r/2}(G) = (-1)^{r/2} \sum_i M_{r/2}^{(i)}(G) \tag{3.2}$$

and in the special case of G a tree, $M_{r/2}(G)$ is the coefficient of $(-\lambda)^{n-r}$ in $\Delta(-\lambda)$ for $A(D)$. The importance of matchings lies in the fact that if G_s is an s-block then the number of conditions imposed by a required mass spectrum for the spin s is $\beta_i(G_s)$, and so can be determined by finding a MM on G_s (Cox 1974a, b, c). However, there is no guarantee that these conditions can be satisfied. It may be impossible to choose the elements of the s-block such that a specified mass spectrum can be realised. The object of this paper is to study this aspect more deeply and develop graph theoretical methods for analysing such mass spectra conditions. We now give a number of results which are used in this approach.

If a graph G contains an edge (v_i, v_j) connecting two otherwise disjoint subgraphs G_1 and G_2 then (v_i, v_j) is called an *isthmus* of G.

Theorem 1. If G is a graph of n vertices with an isthmus (v_i, v_j) connecting subgraphs G_1, G_2 and D, D_1, D_2 the corresponding digraphs then

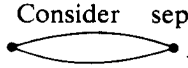
$\mathscr{C}_r(D) = \mathscr{C}_r($  $)$

$$= \sum_{k=0}^{r} \mathscr{C}_k(D_1)\mathscr{C}_{r-k}(D_2) - a_{ij}a_{ji} \sum_{k=0}^{r-2} \mathscr{C}_k(D_1 - v_i)\mathscr{C}_{r-k-2}(D_2 - v_j) \tag{3.3}$$

where $D_1 - v_i$ $(D_2 - v_j)$ denotes the digraph D_1 (D_2) with the vertex v_i (v_j) and all adjacent edges removed, and we define, for any digraph D:

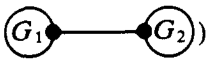
$$\mathscr{C}_0(D) = 1$$

$$\mathscr{C}_s(D) = 0 \qquad \text{if } s < 0 \text{ or } s > n.$$

Proof. Consider separately those SDC which do not contain the 2-cycle  and those that do, using the definition (3.1).

(3.3) is particularly useful when $D_1 = D_2$. Theorem 1 can also be extended to handle a graph with any number of isthmuses—in particular trees—in an obvious way.

For matchings we have the corresponding result

$M_{r/2}($  $)$

$$= \sum_{k=0}^{r}{}' M_{k/2}(G_1)M_{(r-k)/2}(G_2) - \omega_{ij} \sum_{k=0}^{r-2}{}' M_{k/2}(G_1 - v_i)M_{(r-k-2)/2}(G_2 - v_j). \tag{3.4}$$

Here Σ' denotes summation only over even values of k. (3.4) is proved as for theorem 1, but replace SDC by matchings and (3.1) by (3.2).

Using the above results we can often split $\mathscr{C}_r(D)$ into expressions involving $\mathscr{C}_s(D_i)$ where D_i are simple subgraphs of D and s is fairly small, and similarly for $M_{r/2}(G)$. This corresponds to some algebraic organisation of the terms in the coefficients $\mathscr{C}_r(D)$ of $(-\lambda)^{n-r}$ in $|A(D)-\lambda I| = 0$—an organisation based on the structure of the graph G—which often is very useful in the analysis of the conditions imposed by a given mass spectrum.

We can also deduce the possible forms of the minimal polynomial of an associated matrix of a bigraph, and in particular the s-blocks, by direct inspection of the graph G. For high-spin theories, the minimal polynomials we will be interested in will have one of the forms

$$m(-\lambda) = \begin{cases} (-\lambda)^q \prod_{i=1}^{k} [(-\lambda)^2 - m_i^2] & q \geq 1 \quad (3.5) \\ (-\lambda)^q & q > 1. \quad (3.6) \end{cases}$$

In (3.5) all the m_i are distinct. In each case we will call q the *index of nilpotency*. Now from § 2, the s-block can be written in the form

$$A = \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \tag{3.7}$$

by appropriate numbering of vertices. The characteristic equation of A can then be written

$$|A - \lambda I| = \begin{vmatrix} -\lambda I & B \\ B & -\lambda I \end{vmatrix} = |\lambda^2 I - B^2| = |\lambda I - B|\,|\lambda I + B| = 0.$$

So the eigenvalues of A are $\pm\alpha$ where α is an eigenvalue of B. Now if A has the minimum polynomial (3.5) then we can take the eigenvalues of B as $0, m_1, m_2, \ldots, m_k$ and the minimal equation of B will be of the form

$$B^p \prod_{i=1}^{k} (B - m_i) = 0.$$

By substitution of (3.7) into the minimal equation of A we find $p = q$, so the minimal equation of B must be

$$B^q \prod_{i=1}^{k} (B - m_i) = 0. \tag{3.8}$$

Thus,

$$\text{Rank}(B) \geq k + q - 1. \tag{3.9}$$

From a known result of graph theory (Ore 1962, theorem 7.7.2) we have

$$\mathrm{Rank}(B) \leqslant \beta_1(G).$$

This result applies whether or not the MM is perfect. However, if the MM is perfect we can in fact assert

$$\mathrm{Rank}(B) \leqslant \beta_1(G) - 1$$

if (3.8) is to be satisfied, since B is to be singular and if the MM is perfect $\beta_1(G) =$ size of B. So we can state

$$\mathrm{Rank}(B) \leqslant \begin{cases} \beta_1(G) & \text{if MM not perfect} \\ \beta_1(G) - 1 & \text{if MM perfect} \end{cases} \tag{3.10}$$

and with (3.9) this gives

$$q \leqslant \begin{cases} \beta_1 - k + 1 & \text{if MM not perfect} \\ \beta_1 - k & \text{if MM perfect.} \end{cases} \tag{3.11}$$

If A is to be nilpotent, i.e. have minimal polynomial (3.6) then the same arguments yield for the index of nilpotency:

$$q \leqslant \begin{cases} \beta_1 + 1 & \text{if MM not perfect} \\ \beta_1 & \text{if MM perfect.} \end{cases} \tag{3.12}$$

The above gives a graphical method for finding an upper bound on the degree of the minimal polynomial of an s-block A with k particle–antiparticle pairs by direct inspection of the graph.

The graphs also provide a lower bound for the degree of the minimal polynomial. Consider all those pairs of vertices $\{v_i, v_j\}$ which have exactly one path between them, and let d_u be the length of the longest of such paths. Thus, d_u is the length of the longest unique path in the graph. Now for an associated matrix A, the ijth entry in A^l is the sum over edge weight products of all edge sequences between vertices v_i and v_j. It follows that if v_i and v_j are of any two vertices in the graph G separated by d_u, then $A^0, A^1, A^2, \ldots, A^{d_u - 1}$ all have zero in the (i, j) position, while A^{d_u} has a single term which cannot be zero. Thus, the matrices $A^0, A, A^2, \ldots, A^{d_u}$ are linearly independent and the minimum possible degree of the minimal polynomial is $d_u + 1$ (Biggs 1974). This is independent of the mass spectra.

We can now summarise the above results in the following theorem.

Theorem 2. Let A be an associated matrix of the form $\left[\begin{smallmatrix} 0 & B \\ B & 0 \end{smallmatrix}\right]$ of a bigraph $G(V_1, V_2)$, with $|\mathrm{MM}| = \beta_1$ and longest unique path of length d_u. If A has $2k$ non-zero eigenvalues $\pm m_i$, then it has the minimal equation

$$A^q \prod_{i=1}^{k} (A^2 - m_i^2) = 0$$

where

$$d_u - 2k + 1 \leqslant q \leqslant \begin{cases} \beta_1 - k + 1 & \text{if MM not perfect} \\ \beta_1 - k & \text{if MM perfect.} \end{cases}$$

In the particular case of A nilpotent this gives for the nilpotency index:

$$d_u+1 \leqslant q \leqslant \begin{cases} \beta_1+1 & \text{MM not perfect} \\ \beta_1 & \text{MM perfect.} \end{cases}$$

In the case of just two non-zero eigenvalues it reads

$$d_u-1 \leqslant q \leqslant \begin{cases} \beta_1 & \text{MM not perfect} \\ \beta_1-1 & \text{MM perfect.} \end{cases}$$

For the graphs we have to deal with, both d_u and β_1 are easily found by inspection, allowing a quick determination of the possible minimal polynomials of the s-blocks.

A further useful result, applicable if RIR are not present, can be obtained from a recent criterion for irreducibility of (2.1) (Sudarshan *et al* 1977). The system (2.1) is said to be irreducible if it is not covariantly reducible to two or more smaller systems of equations.

Theorem 3. The equation (2.1) based on a representation with non-repeated irreducible representations of $\mathscr{L}_p$ and with associated undirected graph G, is irreducible if and only if G is strongly connected.

When RIR occur the situation becomes more complicated. Note that theorem 3 does not mean the s-blocks must be connected.

4. Analysis of s-block spectra and possibility of unique mass–spin equations without RIR

Consider first the maximum spin block graph G_j, which will be some subgraph of a graph of the form typified in figure 2. We call such graphs *symmetric combs.* Some of the edges may be missing, but we adopt a standard numbering system for the edges, exemplified in figure 2, which is maintained even when edges are missing. As the graph is a tree we need only consider SDC of 2-cycles in D_s, or matchings in G_s. In the latter case we here denote the weight of edge i by a_i. By symmetry of the graph we only have to number the middle edge and one half of the comb. The horizontal edges are numbered odd and the vertical edges are even. The length of the spine of the comb (7 in this case) is called the *width* of the comb, denoted ω, and unless otherwise stated we will assume that the spine is connected—i.e. there are no gaps in it—while vertical edges ('teeth') may be missing. With this notation we can specify all such

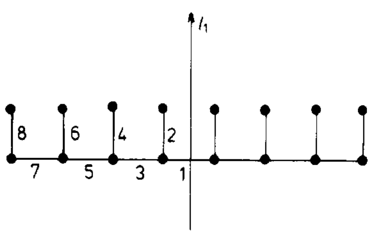


Figure 2. Symmetric comb graph for a maximum spin-$\frac{7}{2}$ block.

combs by the notation

$$(\omega|e_1, e_2, \ldots, e_s) \quad (4.1)$$

where the $e_1, e_2, \ldots, e_s$ are a selection of even numbers chosen from 0 (no teeth) to $\omega+1$.

Theorem 4. A connected symmetric comb cannot have a nilpotent associated matrix.

Proof. First, note that $\beta_1 \leqslant \omega+1$ if edge $\omega+1$ is present and $\beta_1 \leqslant \omega-1$ if not. But if $\omega+1$ present, $d_u = \omega+2$ and theorem 2 gives for the nilpotency index q of any nilpotent associated matrix A:

$$\omega+3 \leqslant q \leqslant \omega+2$$

which is impossible. If $\omega+1$ is not present, $d_u = \omega$ and theorem 2 gives

$$\omega+1 \leqslant q \leqslant \omega$$

again impossible. Hence the associated matrix A cannot be nilpotent.

Further, it can only carry just two non-zero eigenvalues, i.e. correspond to a particle with unique mass, if it is of a certain type (see the following theorem).

Theorem 5. The only symmetric combs which can have an associated matrix with exactly two non-zero eigenvalues are of the form

$$(\omega|2, 4, \ldots, \omega-3, \omega-1).$$

Proof. The minimal equation of the associated matrix would have to have the form

$$A^q(A^2-m^2)=0$$

where from theorem 2

$$d_u - 1 \leqslant q \leqslant \beta_1$$

or

$$d_u \leqslant \beta_1 + 1.$$

If edge $\omega+1$ is present this can only happen for the comb $(\omega|2, 4, \ldots, \omega+1)$—i.e. the combs with no missing teeth. But this comb has a unique perfect matching and so A could not be made singular. So the edge $\omega+1$ must be absent. Then $d_u = \omega$ and we must therefore have $\beta_1 \geqslant \omega - 1$, which can only occur for the comb $(\omega|2, 4, \ldots, \omega-1)$. We would then have $q = \omega - 1$ and so the minimal equation would be

$$A^{\omega-1}(A^2-m^2)=0. \quad (4.2)$$

To verify that the weights of $(\omega|2, 4, \ldots, \omega-1)$ can indeed be chosen so that the associated matrix has just two non-zero eigenvalues, we examine the coefficients $M_{r/2}(G)$ in the characteristic equation, using (3.4), treating edge 1 as isthmus.

In $(\omega|2, 4, \ldots, \omega-1)$ there are 2ω vertices and so the characteristic polynomial of A has the form

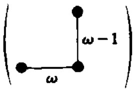
$$\Delta(-\lambda) = (-\lambda)^2((-\lambda)^{2\omega-2} + M_1(G)(-\lambda)^{2\omega-4} + M_2(G)(-\lambda)^{2\omega-6} + \ldots + M_{\omega-1}(G)). \quad (4.3)$$

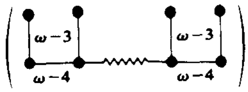
By inspection we obtain

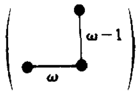
$$M_{\omega-1}(G) = (-1)^{\omega-1}(a_{\omega-1} + a_\omega)^2 a^2_{\omega-3} a^2_{\omega-5} \ldots a^2_2$$
$$= 0 \qquad \text{if } a_{\omega-1} + a_\omega = 0. \tag{4.4}$$

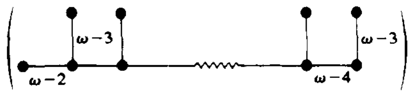
Since $M_{\omega-2}(G)$ must also contain the factor $a_{\omega-1} + a_\omega$, this will also vanish identically on assuming (4.4). We now note the reduction formula:

$M_{\omega-3}((\omega|2, 4, \ldots, \omega-1))$

$= M_1^2$ 

$M_{\omega-5}$ 

$+ 2M_1$ 

$M_{\omega-4}$ 

$+ M_{\omega-3}((\omega-2|2, 4, \ldots, \omega-3))$.

Using (4.4) this gives

$M_{\omega-3}((\omega|2, 4, \ldots, \omega-1))$

$$= M_{\omega-3}((\omega-2|2, 4, \ldots, \omega-3))$$
$$= (-1)^{\omega-3}(a_\omega-2 + a_{\omega-3})^2 a^2_{\omega-5} \ldots a^2_2 \tag{4.5}$$
$$= 0 \qquad \text{if } a_{\omega-2} + a_{\omega-3} = 0.$$

Continuing in this way we obtain

$$M_{r-1}((\omega|2, 4, \ldots, \omega-1)) = 0 \qquad \text{if } a_r + a_{r-1} = 0, r = 3, 4, \ldots, \omega \tag{4.6}$$

and

$$M_1(G) = a_1. \tag{4.7}$$

So, choosing the elements of the associated matrix to satisfy (4.6), we can indeed obtain the characteristic polynomial

$$\Delta(-\lambda) = (-\lambda)^{2\omega-2}(\lambda^2 - a_1) \tag{4.8}$$

as required. The minimal equation will then be

$$A^{\omega-1}(A^2 - a_1) = 0. \tag{4.9}$$

This completes the proof.

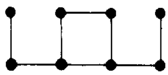
Theorem 5 and its proof tell us exactly what graphs are available for the maximum spin block (which by theorem 4 must be massive), namely the combs $(\omega|2, 4, \ldots, \omega-1)$; furthermore it tells us how to achieve the required mass spectrum, by satisfying (4.6), and that the minimal equation must be of the form (4.9). If the maximum spin block is allowed to have more than one particle–antiparticle pair then we can repeat the analysis using the general form of theorem 2.

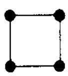
Now that we know the possible connected graphs available for the maximum spin blocks, we have to look at the lower spin blocks. For a unique mass theory these must all be nilpotent. Note that except for the case (1|0), the unique mass maximum spin block cannot be diagonalisable, since by (4.9) the zero eigenvalue factor must be repeated. Further, none of the other blocks can be diagonalisable since they must be nilpotent. Thus, we are dealing with theories in which no s-block can be diagonalised, except in the case $(1|0)$ in which only the maximum spin block can be diagonalised. For such theories the graphs below the maximum spin block can, on the face of it, be quite arbitrary (Amar and Dozzio 1972a, b). However, we will find that the possibilities are in fact very limited.

To continue in the general case, looking at lower-spin blocks, is complicated, so we confine ourselves here to the example of a theory with maximum spin-$\frac{5}{2}$. Then, as we have already seen, the $\frac{5}{2}$ block must be massive (we assume there is in fact a spin-$\frac{5}{2}$ state) and can only be one of $(1|0)$, $(3|2)$ or $(5|2, 4)$. We must now look at $A_{3/2}$ and $A_{1/2}$ which must both be nilpotent for a unique mass–spin theory.

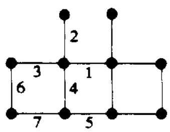
First consider $A_{3/2}$, in each of the three cases for the $\frac{5}{2}$ block:

(i) $(1|0)$. $G_{3/2}$ must be some subgraph of

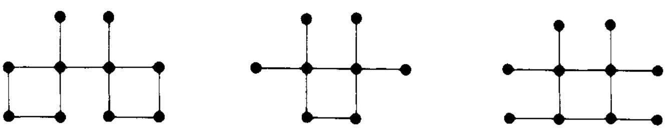


Now so long as the two outside vertical edges are present we have $d_u > \beta_1$ and so by theorem 2 the associated matrix cannot be nilpotent. In fact the only possible nilpotent subgraph in this case is simply 

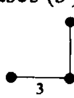
(ii) $(3|2)$. $G_{3/2}$ must be some subgraph of



which contains the edges 1 2 3, and is nilpotent. To ensure $d_u \leq \beta_1$ one or both of 5 or 7 must be present. The only possibilities which are not obviously non-nilpotent (e.g. do not possess a unique M.M) are

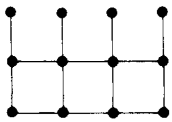


(*a*) (*b*) (*c*)

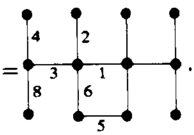
In the case (*a*) a quick consideration of the possible consequent $G_{1/2}$ shows that with such a $G_{3/2}$ these could never be nilpotent (there is no way to avoid a unique MM in $G_{1/2}$), so (*a*) can be eliminated. In both cases (*b*) and (*c*) a maximal SDC in $D_{3/2}$ is a MM in $G_{3/2}$ and yields for nilpotency M_1 

$^2 = 0$. Now from (4.6) a similar relation involving the two edges 2 and 3 is required for unique mass in $A_{5/2}$ and it can be verified that these two equations are in fact incompatible—because of the different $\rho(s, n)$ factors of (2.2) occurring in the different s-blocks. This is an example of the general point that the conditions imposed in different s-blocks must be checked for compatibility. So finally, there are no possibilities for unique mass theories with $G_{5/2} = (3|2)$.

(iii) $(5|2, 4)$. $G_{3/2}$ must be a subgraph of

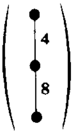


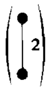
and in fact the only possibility that does not spoil $G_{1/2}$ or is not obviously non-nilpotent is

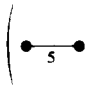
$G_{3/2} =$ 

.

Now

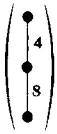
$\mathscr{C}_{10}(D_{3/2}) = \mathscr{C}_2$ 

$\mathscr{C}_2^2$ 

$\mathscr{C}_2$ 

$= 0$

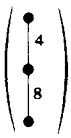
whence

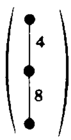
$\mathscr{C}_2$ 

$= 0.$

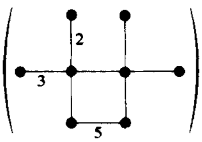
This also ensures

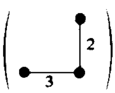
$$\mathscr{C}_8(D_{3/2}) = 0$$

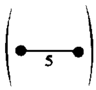
since $\mathscr{C}_2$ 

is clearly a factor of $\mathscr{C}_8(D_{3/2})$. Now, again using $\mathscr{C}_2$ 

$= 0$,

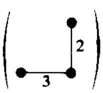
$\mathscr{C}_6(D_{3/2}) = \mathscr{C}_6$ 

$= \mathscr{C}_2^2$ 

$\mathscr{C}_2$ 

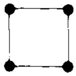
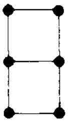
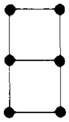
$= 0$

which gives

$\mathscr{C}_2$ 

$= 0$

and again this conflicts with the corresponding equation of (4.6) for $A_{5/2}$. Thus $(5|2, 4)$ also fails to provide a good spin-$\frac{3}{2}$ block.

From the above we see that the only possible $G_{5/2}$ is and only possible $G_{3/2}$ is . It is then immediate that the only possible $G_{1/2}$ is . Thus, the only possible unique mass–spin-$\frac{5}{2}$ theory without RIR must be based on the graph $G =$ . However, as Capri (1969) has noted, this graph cannot lead to a unique mass theory because in fact on closer analysis the conditions for nilpotency of $A_{3/2}$ and $A_{1/2}$ are incompatible. We thus finally conclude that there are no unique mass–spin-$\frac{5}{2}$ theories without RIR. This is despite considering all possible available irreducible representations of $\mathscr{L}_{\mathrm{p}}$ (Capri for example only considered a very restricted subset of these—in fact the central tower of unit width). We conjecture that this is so for all spins-$\frac{5}{2}$ or greater and so for these we must introduce RIR. Note that for spin-$\frac{3}{2}$ there is in fact precisely one spin-$\frac{3}{2}$ theory with unique mass, without RIR, and this is based on the graph containing the representations $\mathscr{D}(\frac{1}{2}\ 1\ \frac{1}{2}) \oplus \mathscr{D}(\frac{1}{2}\ 0\ \frac{1}{2})$. This theory is equivalent to the usual Fierz–Pauli or Rarita–Schwinger spin-$\frac{3}{2}$ theory. The situation for integer spin theories is not so clear, but we would be surprised if similar results did not follow.

The above work was based on the assumption that $G_{5/2}$ was connected. We now show that this is in fact necessary for a unique mass–spin-$\frac{5}{2}$ theory. By an obvious extension of the corresponding integer spin result given in Cox (1974a, b, c), $G_{5/2}$ must have the form shown in figure 3 where G_1 is connected and non-trivial, while G_2

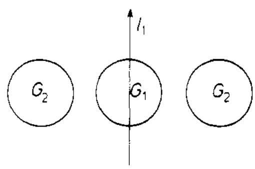
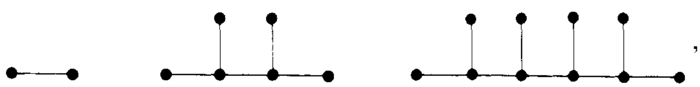
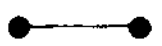
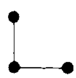


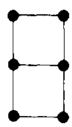
Figure 3. General form for a disconnected maximum spin block. G_1 must be non-trivial.

can be disconnected combs, which must have nilpotent matrices. From the previous work the only candidates for G_1 are

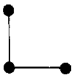


of which only  then allows nilpotent G_2, namely, . Thus we can only have

$G_{5/2} =$ .

But for this graph the $D_{3/2}$ and $D_{1/2}$ are identical to the previous case of $G =$ 

and we know these cannot be both nilpotent—in short the addition of the extra edges



makes no difference to the nilpotency of $A_{3/2}$, $A_{1/2}$.

5. Equations with RIR

Since we seem forced into using RIR we must consider how much of our graph theory carries over to this case. The answer is, most of it. The strongest restrictions placed on our graphs were that the corresponding undirected graphs be bipartite and that the corresponding digraphs be symmetric about the l_1 axis. In fact, even when arbitrary numbers of repeated representations are introduced the graphs are still bipartite and symmetric. They are however no longer lattice graphs and indeed not even planar in general, so the graph theory results become more difficult to apply because of the consequent loss in pictorial simplicity. Also, theorem 3 does not apply.

As a first example, consider the representation

$$= 2(-\tfrac{1}{2}, \tfrac{3}{2}) \oplus 2(\tfrac{1}{2}, \tfrac{3}{2}) \oplus (-\tfrac{1}{2}, \tfrac{5}{2}) \oplus (\tfrac{1}{2}, \tfrac{5}{2}) \tag{5.1}$$

which has been studied in detail by Hurley and Sudarshan (1975). The graph G and the s-block graphs G_s are shown in figure 4. $A_{3/2}$ can clearly be made massive.

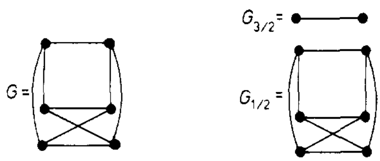


Figure 4. G and G_s graphs for the representation (6.1).

Further, if $A_{1/2}$ is to be nilpotent theorem 2 shows that the index of nilpotency must satisfy $2 \leqslant q \leqslant 3$ since from the graph $d_u = 1$, $\beta_1 = 3$ with a perfect matching. This implies that the minimal equation of L_0 must be either

$$L_0^2(L_0^2 - m^2) = 0$$

or (5.2)

$$L_0^3(L_0^2 - m^2) = 0$$

which is precisely what Hurley and Sudarshan found. Of course, their calculations were more lengthy because they actually exhibited the form of L_0 and the above equations, while our results are merely a list of possibilities which have to be checked by the further detailed calculations. However, as a first guide the graphical approach is useful, and as we have seen can also organise and simplify the necessary calculations.

Recently Singh and Hagen (1974) have generalised the Rarita–Schwinger Lagrangian theory to arbitrary spin using the representation

$$= (-\tfrac{1}{2}, s+1) \oplus (\tfrac{1}{2}, s+1) \oplus (-\tfrac{1}{2}, s) \oplus (\tfrac{1}{2}, s)$$

$$\oplus 2 \sum_{l=1/2}^{s-2}{}_{\oplus} ((-\tfrac{1}{2}, l+1) \oplus (\tfrac{1}{2}, l+1))$$

in (l_0, l_1) notation. This choice was governed by the requirement of minimum number of field components. If it is true, as our results suggest, that RIR are essential for spins-$\frac{5}{2}$ or greater then the Singh and Hagen representation would seem to be the most natural to adopt for high-spin theories, because of the economy in the number of field components and the comparatively simple graphical form. (Note however that the Singh–Hagen theory is almost certainly acausal, simply because its spin-$\frac{3}{2}$ case is.)

Consider for example the Singh–Hagen representation for spin-$\frac{5}{2}$. The relevant graphs are given in figure 5. This representation has also been studied by Capri (1969)

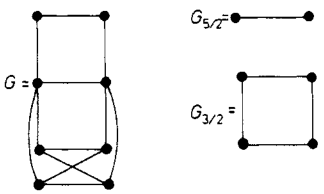


Figure 5. G and G_s graphs for the Singh–Hagen spin-$\frac{5}{2}$ theory.

and Frank (1973). By inspection of the G_s graphs we see that the required conditions for unique mass can be satisfied—$A_{5/2}$ can be made massive and $A_{3/2}$, $A_{1/2}$ can be made nilpotent, all independently. By noting d_u and β_1 for $G_{3/2}$ and $G_{1/2}$ we see that the minimal polynomial of $A_{3/2}$ must be $A_{3/2}^2=0$, and that of $A_{1/2}$ can only be $A_{1/2}^q=0$ where $3\leq q\leq 4$. This means that the minimal equation for L_0 can at most only be one of

$$L_0^3(L_0^2-m^2)=0$$

or

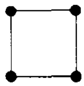
$$L_0^4(L_0^2-m^2)=0.$$

In fact, by generalising the above we find that for the general Singh–Hagen representation for half odd integer spin-j, the minimal equation of L_0 can only be at most one of

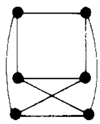
$$L_0^q(L_0^2-m^2)=0 \qquad j-\tfrac{1}{2}\leq q\leq 2j-1. \tag{5.3}$$

However, from standard work of Umezawa and Visconti (1956) corrected by Glass (1971) and Chandrasekaran *et al* (1972) we must have $q\geq 2j-1$, where j is the maximum physical spin actually carried by the field (not the maximum spin in the representation $\mathscr{R}$, although in the Singh–Hagen theory they *are* the same thing). Thus, there must be at least one s-block in the theory which has nilpotency index $2j-1$ or greater. By considering MM on the s-blocks of the general Hagen and Singh representation for maximum spin-j, only the $\frac{1}{2}$ block can satisfy this, and from (6.3) the minimal equation of L_0 can only be

$$L_0^{2j-1}(L_0^2-m^2)=0. \tag{5.4}$$

In particular, the Hagen–Singh theory cannot suffer from the Glass pathology in which $q>2s-1$. From the graphs we can easily see how this pathology can occur. All we have to do is to add another repeated representation below the maximum spin-j. This will increase β_1 without increasing the spin and so by theorem 3 will allow the nilpotency index of the $\frac{1}{2}$ block to rise above $2j-1$. An example of this is provided by the Hurley–Sudarshan representation (5.1). The original Rarita–Schwinger spin-$\frac{3}{2}$ representation is based on the graph  for which $\beta_1(G_{1/2})=2$, yielding nilpotency index 2 for the $\frac{1}{2}$ block, and therefore minimal equation

$$L_0^2(L_0^2-m^2)=0$$

for L_0, which fits in with '$q=2j-1$'. However, on doubling up the $\mathscr{D}(\frac{1}{2}\ 0\ \frac{1}{2})$ representation the graph becomes  for which $\beta_1(G_{1/2})=3$, yielding the possible minimal equations (5.2). The higher degree equation violates the original Umezawa–Visconti relation, and indeed is precisely the counter example used by Glass (1971). Note that the condition that in a theory with maximum physical spin-j there must be at least one s-block with nilpotency index $2j-1$ or greater can be translated into the graphical condition that at least one G_s must have a $\beta_1\geqslant 2j-1$. Although we have not done so here, this could clearly be used in eliminating possibilities, in addition to the methods of § 4. For example, it immediately confirms the statement of Capri (1969) that the central tower of unit width, with no RIR, cannot support a unique mass theory for spin-$j\geqslant\frac{3}{2}$. This follows because $\beta_1=s+\frac{1}{2}$ for the representation, so $\beta_1\geqslant 2j-1$ gives $j\leqslant\frac{3}{2}$.

6. Conclusion

The problem of high-spin equations coupled to an external electromagnetic field needs little introduction. At the quantised level the field commutators involve the external field and violate relativistic invariance (Johnson and Sudarshan 1961), and at the classical level this is manifested as acausal (or failure of) propagation for certain values of the external field (Velo and Zwanzinger 1969). It seems now that all well known high-spin theories have been tested for the Velo–Zwanzinger acausality pathology and the only theories free of acausality are those requiring indefinite metric for quantisation—for example Bhabha's multi-mass equations. There is a tendency therefore to suspect that acausality is inevitable for spin greater than 1 unless an indefinite metric quantisation is accepted. However, this suspicion is really based on a very small number of examples, and there are apparently vast numbers of possible free field theories for a given spin—there is no general proof that they will all be acausal when we turn on the external field. It is therefore important to find some way of classifying and studying the possible free field theories, so that one has an idea of what is available for use in an interaction theory. The testing of these theories for consistency is another problem. The consistency problems arise essentially from the constraints in the theory, which in turn are related to the minimal polynomials of L_0 and the s-blocks. It is therefore useful to have a quick method of finding the possible minimal polynomials also.

We have presented a graph theoretical approach to the problem of field equations based on reducible representations of $\mathscr{L}_p$ corresponding to half integer spin which should help in locating good free field theories. The advantage of the approach is particularly apparent in the absence of RIR, for then the graphs are very simple, and a great deal can be said about the properties of the equations without any algebraic calculation at all. Having given a number of relevant results we have applied these to show the very limited range of possible theories available for describing unique mass–spin if we do not use RIR. In the case of a wider mass–spin spectrum the methods are still applicable, but then the quantisability of the resulting theory must also be tested (which in fact is also assisted by use of the special form (2.3) for the s-blocks). In previous attempts at constructing high-spin unique-mass theories few people have taken advantage of all available representations in the Fermi-plane but have chosen to introduce extra field components by introducing RIR. The work of § 4 suggests that this may in any case be essential for half odd integers of $\frac{3}{2}$ or more, and we would expect a similar result for integer spin. When RIR are allowed the graph theoretical methods still apply, but are more difficult to use because of the complexity of the graphs. Nevertheless, they provide a useful initial approach to finding good theories.

A number of authors have considered alternative graph theoretical approaches to high-spin equations.

Shelepin (1961) used standard graphical methods of angular momentum analysis to study the structure of the L_μ *algebra*, while Biritz (1975a, b) has applied the same techniques to the L_μ matrices in the special case of L_0 Hermitian. Despite well developed techniques, the angular momentum approach is very complicated, even for the simple case of the Fierz–Pauli spin-$\frac{3}{2}$ algebra (Shelepin 1961), although the analysis of such algebras by any means is notoriously difficult. (The Fierz–Pauli spin-$\frac{3}{2}$ theory is of course not included in Biritz's analysis, since L_0 is not Hermitian in this case.) Only in the case of L_0 Hermitian does it seem that the angular momentum graphical analysis is a convenient tool for high-spin analysis, as developed by Biritz.

In the graph theoretical approach presented in this paper there is no need for angular momentum analysis of the spin structure, because the mass spectra for each spin block are considered separately. This is the specific advantage of the Gel'fand–Yaglom representation—it is adapted to the space rotation subgroups of the representation $\mathcal{R}$. From a group theoretical point of view the graphs of this paper represent connectivities implied by the generalised Clebsch–Gordon theorem applied to $\mathcal{D}(\frac{1}{2}\,\frac{1}{2})\oplus\mathcal{R}$, in a representation of L_0 adapted to the space rotation groups in $\mathcal{R}$, while the angular momentum graphs represent connectivities implied by the Clebsch–Gordon theorem applied to

$$\Big(\sum_{i}{}_{\oplus}\,\mathcal{D}(i)\Big)\oplus\Big(\sum_{j}{}_{\oplus}\,\mathcal{D}(j)\Big)$$

where

$$\sum_{i}{}_{\oplus}\,\mathcal{D}(i)$$

is the rotation group decomposition of $\mathcal{R}$.

Another graphical approach to high-spin equations is that of Frank (1973). He uses essentially the same graphs as the present paper, but only to characterise and represent the Lagrangian theories, and to represent decomposition into s-blocks. He makes no use of graph theoretical methods to actually examine the structure of specific theories, or to assist in calculations.

We remark finally that the essential features of much of this work are the simple bipartite form of the graphs involved, and the reflection symmetry. Ultimately these depend on the structure of the group under which the equations are covariant—the

Lorentz group in this case. Thus, the graph theoretical approach is merely a pictorial means of taking advantage of the high degree of symmetry imposed on the equations by the group structure. One would therefore expect the same sort of device to be useful for the practical study of equations covariant under other groups.

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